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On heat and mass transfer in a viscoelastic boundary layer flow over an exponentially stretching sheet

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Abstract

A comprehensive mathematical analysis has been carried out on momentum, heat and mass transfers in a viscoelastic boundary layer fluid flow over an exponentially stretching continuous sheet. The contributions of the viscous dissipation and elastic deformation have been taken into account in this study. Zeroth order analytical local similar solution of the highly non-linear stream function equation and confluent hypergeometric solutions of heat and mass transfer equations are obtained. These solutions involve an exponential dependence of stretching velocity, prescribed boundary temperature and concentration, prescribed boundary heat flux and concentration flux on the flow directional coordinate. The contribution of elastic deformation on various heat transfer characteristics is examined. The effects of various physical parameters like local viscoelastic parameter, Prandtl number, local Reynolds number, local Eckert number and Schmidt number on various momentum, heat and mass transfers characteristics are presented in this paper.

Keywords: Viscoelastic fluid; Boundary layer flow; Exponentially stretching surface; Heat and mass transfers and elastic deformation

1. Introduction

A great deal of works has been carried out on various aspects of momentum and heat transfer characteristics in a viscoelastic boundary layer second-order fluid flow over a stretching plastic boundary [1–3] since the pioneering work of Sakiadis [4]. Qualitative analysis of these studies have significant bearing on several industrial applications such as polymer sheet extrusion from a dye, glass fiber and paper production, drawing of plastic films etc. Most of the available literature deals with the study of viscoelastic boundary layer flow over a stretching surface where the velocity of the stretching surface is assumed linearly proportional to the distance from the fixed origin. However, it is often argued that [5] realistically stretching of plastic sheet may not necessarily be linear. This situation was beautifully dealt by Kumaran and Ramanaiah [6] in their work on boundary layer fluid flow where, probably first time, a general quadratic stretching sheet has been assumed. They dealt with viscous fluid flow, whereas in reality, most of the fluid flow applications in polymer processing industries are concerned with non-Newtonian fluids.

Another important aspect, that is, heat transfer processes play an important role in all such theoretical studies. This is due to the fact that the rate of cooling influences a lot to the quality of the final product with desired characteristics. In view of this we have presented some works on heat transfer in a viscoelastic boundary layer flow over a linear stretching sheet [7,8]. There are several other researchers who investigated various aspects of heat transfer characteristics over linearly stretching sheets [9–12]. Recently Ali [13] investigated thermal boundary layer by considering a power law stretching surface. A new dimension is added to this investigation by Elbashbeshy [14] who examined the flow and heat transfer characteristics by considering exponentially stretching sheet.

In view of the above we extend the work of Elbashbeshy [14] to viscoelastic fluid flow, heat and mass transfer. Approximate analytical local similar solutions are obtained for stream function and velocity distribution by transforming highly non-linear differential equation into Riccati type and then solving this sequentially. Local similar solutions for temperature and

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Nomenclature

u, v	velocity components	Re	local Reynolds number
k	thermal conductivity	ν_0, A_0	temperature parameters in PEST case
k_0	elastic parameter	ν_1, A_1	concentration parameters in PESC case
μ	is the limiting viscosity at small rate of shear	Pr	Prandtl number
D	molecular diffusivity of chemically reactive species	E	local Eckert number
k_c	first-order chemical reaction	Sc	Schmidt number
U_0	characteristic velocity	r_c	local rate of first-order chemical reaction
l	characteristic length	Pr^*	modified Prandtl number
k_1^*	local viscoelastic parameter	Sc^*	modified Schmidt number
η	pseudo-similarity variable	E_2, ν_2	temperature parameters in PEHF case
C_f	local skin-friction coefficient	E_3, ν_3	concentration parameters in PEMF case

concentration are obtained in the form of confluent hypergeometric function for non-isothermal boundary conditions of both the types (1) prescribed exponential order surface temperature (PEST) and (2) prescribed exponential order boundary heat flux (PEHF). In the present paper we analyse the effect of various physical parameters like local viscoelastic parameter, Prandtl number, local Reynolds number, Schmidt number and local Eckert number on various momentum, heat and mass transfer characteristics of boundary layer second-order fluid flow over an exponentially stretching continuous surface.

2. Formulation of the problem

We consider two-dimensional steady-state boundary layer flow of incompressible viscoelastic second-order fluid over a stretching sheet for analysis. The flow is assumed to be generated by stretching of the elastic boundary sheet from a slit with a large force such that the velocity of the boundary sheet is an exponential order of the flow directional coordinate x (Fig. 1). We take into account the frictional heating due to viscous dissipation as the fluid considered for analysis is of viscoelastic type which possess viscous property also. We also consider the effect of elastic deformation in energy balance in order to take into account the elastic effect of the viscoelastic fluid. In this situation the governing boundary layer equations for momentum [2] and heat transfer [15] are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2.1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \gamma\frac{\partial^2 u}{\partial y^2} - k_0\left\{\frac{\partial}{\partial x}\left(u\frac{\partial^2 u}{\partial y^2}\right) + \frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial y^2} + v\frac{\partial^3 u}{\partial y^3}\right\} \tag{2.2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{k}{\rho c_p}\frac{\partial^2 T}{\partial y^2} + \frac{\mu}{\rho c_p}\left(\frac{\partial u}{\partial y}\right)^2 - \frac{\delta k_0}{c_p}\left\{\frac{\partial u}{\partial y}\frac{\partial}{\partial y}\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right)\right\} \tag{2.3}$$

where u and v are the velocity components of the fluid in x and y directions respectively, γ is the kinematic coefficient of

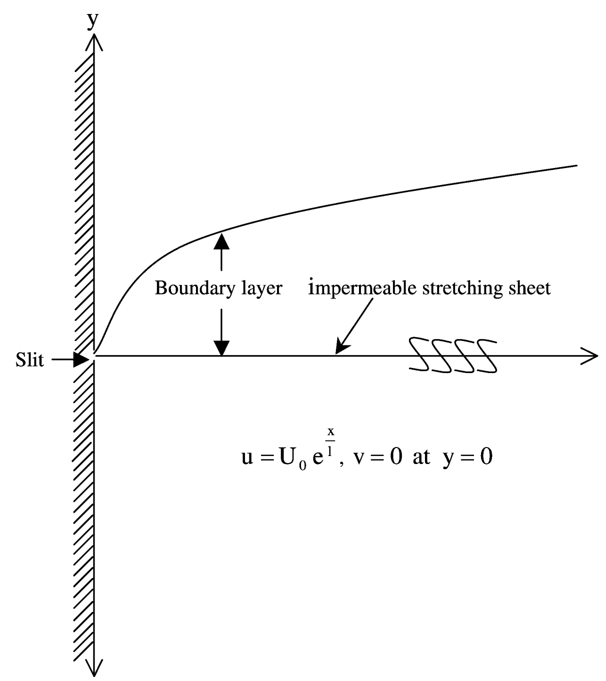


Fig. 1. Boundary layer over an impermeable exponentially stretching sheet.

viscosity, $k_0 = -\alpha_1/\rho$ is the elastic parameter where α_1 is the material modulus and α_1 takes negative value for second-order fluid and k is the thermal conductivity of the fluid medium. In deriving equation (2.2) it is assumed that the normal stress is of the same order of magnitude as that of the shear stress, in addition to usual boundary layer approximations. Eq. (2.3) is the thermal boundary layer equation which takes into account the viscous dissipation and elastic deformation. In order to analysis the effect of the elastic deformation term on the heat transfer we have introduced a term δ. In this regard let us mention that there are several research works on viscoelastic boundary layer flow which do not take into account the elastic deformation [9, 16–19]. In order to make out the contribution of the elastic deformation term to the heat transfer process we have introduced the parameter δ. When $\delta = 0$ our paper results give results of most of the above referred works as limiting cases, although they are not realistic. When $\delta = 1$ we get realistic result.

We assume that the viscoelastic fluid contains some chemically reactive diffusive species. The boundary layer equation governing diffusion of chemically reactive species [20] is

$$u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}=D\frac{\partial^2 C}{\partial y^2}-k_c C^n \tag{2.4}$$

where C is the concentration of the chemically reactive species, D is the molecular diffusivity of chemically reactive species, k_c the rate of chemical conversion and other quantities have their usual meanings.

2.1. Boundary conditions on velocity

In order to take into account the effect of exponential stretching of the boundary surface causing flow along x direction we employ the following boundary conditions on velocity:

$$\begin{aligned} &u=U_w(x)=U_0\,\mathrm{Exp}\left(\frac{x}{l}\right),\ v=0 \quad \text{at } y=0\\ &u=0 \quad \text{as } y\to\infty \end{aligned} \tag{2.5}$$

Here U_0 is a constant and l is the reference length. The above exponential boundary condition is valid only when $x \ll l$ which occurs very nearer to the slit. Following Elbashbeshy [14] we have considered the above three boundary conditions for the problem. To the best of author's knowledge, all the available literature on boundary layer flow of viscoelastic fluid over linearly stretching sheets [1–3,10,17,21,22] deal with three boundary conditions on velocity, which are one less than the number required to solve the equation uniquely. Rajagopal [1] solved the problem with three boundary conditions only using perturbation expansion. Troy et al. [23] gave an exact solution of the problem for linearly stretching boundary conditions. Later Chang [24] discussed non-uniqueness of the problem and derived different non-unique solutions. However, Rao [22], Lawrence and Rao [11] argue that among all the available solutions Troy's solution containing exponential terms is physically realistic as it recovers the Navier–Stokes solution when viscoelastic parameter is taken to be zero. In view of this we present in the next section physically realistic sequential local similar solutions of viscoelastic boundary layer flow problem over an exponentially stretching sheet.

3. Solution of the momentum equation

The solution (2.2) may be written in terms of stream function $\psi(x,y)$ which satisfies the equation of continuity (2.1). Hence, we can write

$$u=\frac{\partial\psi}{\partial y},\qquad v=-\frac{\partial\psi}{\partial x} \tag{3.1}$$

Stream function $\psi(x,y)$ is assumed to be in the form

$$\psi(x,y)=\sqrt{2\gamma l U_0}\,f(x,\eta)\,\mathrm{Exp}\left(\frac{x}{2l}\right) \tag{3.2}$$

$$\eta=y\sqrt{\frac{U_0}{2\gamma l}}\,\mathrm{Exp}\left(\frac{x}{2l}\right) \tag{3.3}$$

Making use of these transformations in Eq. (2.2) we obtain the following partial differential equation

$$\begin{aligned} &2f_\eta^2-ff_{\eta\eta}-f_{\eta\eta\eta}+k_1^*\left(3f_\eta f_{\eta\eta\eta}-\frac{1}{2}ff_{\eta\eta\eta\eta}-\frac{3}{2}f_{\eta\eta}^2\right)\\ &\quad=2lf_x f_{\eta\eta}-2lf_\eta f_{\eta x}\\ &\qquad-k_1^*[lf_\eta f_{\eta\eta\eta x}-lf_x f_{\eta\eta\eta\eta}-lf_{\eta\eta}f_{\eta\eta x}+lf_{\eta x}f_{\eta\eta\eta}] \end{aligned} \tag{3.4}$$

The solution of the above equation would present a local non-similarity solution, which is highly complicated for the present viscoelastic boundary layer problem to be derived in closed form. In view of this we may look for similarity solutions or local similarity solutions. However, all the existing literature shows that if $k_1^*=0$ then only similarity solution exists. In this regard the solutions of Magyari and Keller [25] may be cited here. There are numerous works of Magyari and Keller along with Pop on power law stretching problem with general x dependent stretching velocity, power law stretching and power law shearing problems are [25–30].

In all above problems similarity solutions have been obtained. The fluid considered by them are viscous fluid. In our present problem the presence of viscoelastic term in the momentum boundary layer equation (2.2), dissipation term in thermal boundary layer equation (2.3) and chemical reaction term in concentration boundary layer equation (2.4) do not allow the problem to have the self similar solutions. Hence we turn our attention to obtain local similar solution of the problem by introducing a pseudo-similarity variable. In order to obtain local similarity solution we must have the condition that derivatives of f with respect to x do not exist. Therefore $f(x,\eta)$ should be assumed as $f(\eta)$ i.e., $f(x,\eta)=f(\eta)$. With this assumption RHS of Eq. (3.4) vanishes and we obtain the following fourth-order non-linear quasi-ordinary differential equation of the form

$$2f_\eta^2-ff_{\eta\eta}=f_{\eta\eta\eta}-k_1^*\left[3f_\eta f_{\eta\eta\eta}-\frac{1}{2}ff_{\eta\eta\eta\eta}-\frac{3}{2}f_{\eta\eta}^2\right] \tag{3.5}$$

where $k_1^*=\frac{k_0 U_w}{\gamma l}$ is the dimensionless local viscoelastic parameter and the subscript η stands for differentiation with respect to η.

The corresponding boundary conditions on f are:

$$\begin{aligned} &f=0,\quad f_\eta=1 \quad \text{at } \eta=0\\ &f_\eta=0,\quad \text{as } \eta\to\infty \end{aligned} \tag{3.6}$$

Integrating Eq. (3.5), we obtain

$$\begin{aligned} f_{\eta\eta}+ff_\eta=-S+\int_0^\eta\bigg[3f_{\eta_1}^2+k_1^*\bigg\{&3f_{\eta_1}f_{\eta_1\eta_1\eta_1}\\ &-\frac{1}{2}ff_{\eta_1\eta_1\eta_1\eta_1}-\frac{3}{2}f_{\eta_1\eta_1}^2\bigg\}\bigg]\,\mathrm{d}\eta_1 \end{aligned} \tag{3.7}$$

For $\eta\to\infty$, we get

$$S=\int_0^\infty\left[3f_{\eta_1}^2+k_1^*\left\{3f_{\eta_1}f_{\eta_1\eta_1\eta_1}-\frac{1}{2}ff_{\eta_1\eta_1\eta_1\eta_1}-\frac{3}{2}f_{\eta_1\eta_1}^2\right\}\right]\mathrm{d}\eta_1 \tag{3.8}$$

Higher-order derivatives in Eq. (3.7) can be reduced through integration by-parts. However, as boundary conditions on higher order derivatives are not known, integration constants would be undetermined. Thus, practically integration by-parts of Eq. (3.7) would not be of much use. Consequently Eq. (3.8) does not give exact expression of S_0 until and unless the functional form of $f(\eta)$ is prescribed. Keeping in this mind we integrate Eq. (3.7) once more and use the boundary condition of Eq. (3.6), to get

$$f_\eta + \frac{1}{2}f^2 = 1 - S\eta + \int_0^\eta \left[\int_0^{\eta_2} 3f_{\eta_1}^2 + k_1^* \left(3f_{\eta_1} f_{\eta_1\eta_1\eta_1} - \frac{1}{2} f f_{\eta_1\eta_1\eta_1\eta_1} - \frac{3}{2} f_{\eta_1\eta_1}^2 \right) \mathrm{d}\eta_1 \right] \mathrm{d}\eta_2 \tag{3.9}$$

This non-linear equation may be solved by substituting suitable zeroth-order approximation $f_\eta^{(0)}(\eta)$ for $f_\eta(\eta)$ on the RHS. Hence the solution procedure is reduced to the sequential solution of the Riccati-type equation:

$$f_\eta^{(n)} + \frac{1}{2} f^{(n)^2} = \mathrm{RHS}\left(f_\eta^{(n-1)}, f_{\eta\eta}^{(n-1)}, f_{\eta\eta\eta}^{(n-1)}, f_{\eta\eta\eta\eta}^{(n-1)}\right) \tag{3.10}$$

In order to know the nature of expression of the zeroth-order approximation we depend on the knowledge of numerical solution of Eq. (3.5) employing Runge–Kutta fourth-order method with shooting (Fig. 2). Pattern of the graphs of $f(\eta)$ and $f_\eta(\eta)$ in Fig. 2 and the nature of boundary conditions in Eq. (3.6) have promoted us to assume the zeroth-order approximation of $f_\eta(\eta)$ in the form:

$$f_\eta^{(0)}(\eta) = \mathrm{Exp}(-S_0\eta) \tag{3.11}$$

This solution satisfies the both given boundary conditions on $f_\eta(\eta)$.

Integrating Eq. (3.11) and using boundary conditions of Eq. (3.6) we obtain

$$f^{(0)}(\eta) = \frac{1 - \mathrm{Exp}(-S_0\eta)}{S_0} \tag{3.12}$$

Now we substitute all the derivatives of zeroth-order approximation $f^{(0)}(\eta)$ into RHS of Eq. (3.9) and assume that first-order iteration $f^{(1)}(\eta)$ on the LHS of Eq. (3.9) satisfies the boundary conditions of (3.6). In this process we obtain the expression of S_0 as

$$S_0 = \sqrt{\frac{3}{2(1-k_1^*)}}, \qquad f_{\eta\eta}^{(0)}(0) = -S_0 \tag{3.13}$$

Making use of Eqs. (3.11) and (3.12) in Eq. (3.9) and integrating, we get differential equation for first-order approximation $f^{(1)}(\eta)$ in the form:

$$f_\eta^{(1)} + \frac{1}{2} f^{(1)2} = 1 + \frac{(3 + k_1^* S_0^2)}{4S_0^2}\left(e^{-2S_0\eta} - 1\right) + \frac{k_1^*}{2}\left(e^{-S_0\eta} - 1\right) \tag{3.14}$$

The above is a non-linear Riccati-type equation and this can be solved for $f^{(1)}(\eta)$ analytically. However, we use zeroth-order approximation $f^{(0)}(\eta)$ for solving the energy equation in the next section. Regarding the accuracy of the approximate solution $f^{(0)}(\eta)$ we compare the zeroth-order approximate solution (3.12) with the numerical solution of the momentum equation (3.5) by employing Runge–Kutta fourth-order method with shooting (Fig. 2). Comparison study of these solutions shows that zeroth-order approximate solutions have very good accuracy with the corresponding numerical solutions near the boundary region. In view of this zeroth-order approximate solutions $f^{(0)}(\eta)$ and $f_\eta^{(0)}(\eta)$ are used to solve the heat transfer equation. In addition to this, zeroth-order solution given by Eq. (3.12) would enable us to obtain analytical solution of energy equation in the form of confluent hypergeometric function.

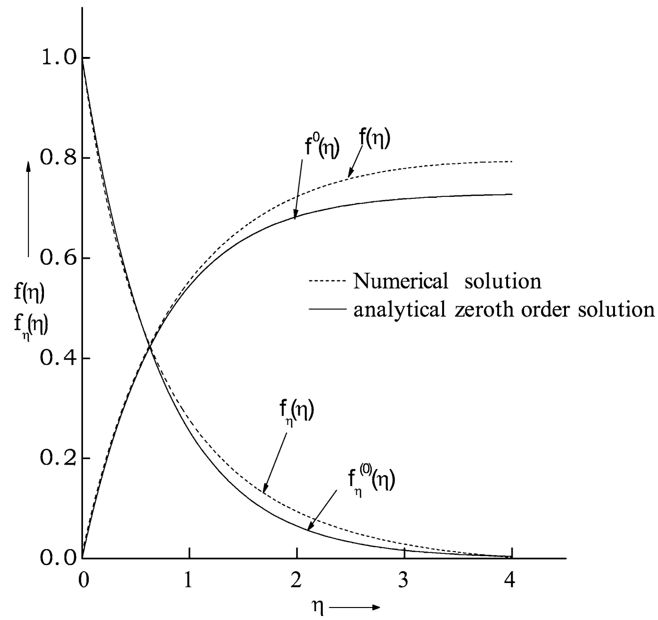


Fig. 2. Profiles of streamfunction $f(\eta)$ and velocity function $f_\eta(\eta)$ obtained from numerical as well as analytical method when $k_1^* = 0.2$.

The dimensionless local skin-friction coefficient C_f is expressed

$$C_f = -\frac{\left(\gamma\frac{\partial u}{\partial y} - k_0\left\{u\frac{\partial^2 u}{\partial x \partial y} - 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right\}\right)}{U_0^2\,\mathrm{Exp}\left(\frac{2x}{l}\right)} \quad \text{at } y = 0$$
$$= \frac{S_0}{\sqrt{2Re}}\left[1 - \frac{7}{2}k_1^*\right] \tag{3.15}$$

where $Re = \frac{U_w l}{\gamma}$ is the local Reynolds number.

4. Solutions of the heat and mass transfer equations

In order to solve the thermal boundary layer equation (2.3) and concentration boundary layer equation (2.4) we consider two general cases for non-isothermal temperature boundary conditions and concentration boundary conditions, namely:

(A) Boundary with prescribed exponential order surface temperature (PEST) and exponential order surface concentration (PESC).
(B) Boundary with prescribed exponential order heat flux (PEHF) and exponential order Mass flux (PEMF).

Case A: Prescribed exponential order surface temperature (PEST) and surface concentration (PESC). In PEST and PESC cases we employ the following surface boundary conditions on temperature and concentration:

$$\begin{aligned} &T = T_w = T_\infty + A_0 \operatorname{Exp}\left(\frac{\nu_0 x}{2l}\right) \quad \text{at } y = 0 \\ &T \to T_\infty \quad \text{as } y \to \infty \\ &C = C_w = C_\infty + A_1 \operatorname{Exp}\left(\frac{\nu_1 x}{2l}\right) \quad \text{at } y = 0 \\ &C \to C_\infty \quad \text{as } y \to \infty, \end{aligned} \tag{4.1}$$

where ν_0 and A_0 are the parameters of temperature and ν_1 and A_1 are parameters of concentration distribution on the stretching surface. T_∞ and C_∞ are the temperature and concentration respectively far away from the stretching sheet.

In order to obtain local similar solution for temperature and concentration we define dimensionless temperature and concentration variables as follows:

$$\theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}, \qquad \phi(\eta) = \frac{C - C_\infty}{C_w - C_\infty} \tag{4.2}$$

where $T_w - T_\infty$ and $C_w - C_\infty$ are given by Eq. (4.1). Using Eq. (4.1) the dimensional energy equation (2.3) and the dimensional concentration equation (2.4) take the following non-dimensional forms:

$$\begin{aligned} &\theta_{\eta\eta} + Pr(f\theta_\eta - \nu_0 f_\eta \theta) \\ &\quad = -Pr\,E\left[f_{\eta\eta}^2 - \frac{\delta k_1^*}{2} f_{\eta\eta}\{3 f_\eta f_{\eta\eta} - f f_{\eta\eta\eta}\}\right] \end{aligned} \tag{4.3}$$

$$\phi_{\eta\eta} + Sc\,f\phi_\eta - Sc(\nu_1 f_\eta + r_c)\phi \tag{4.4}$$

where $Pr = \frac{\gamma}{\alpha}$ is the Prandtl number, $\alpha = \frac{\kappa}{\rho c_p}$ is the thermal diffusivity of the fluid, $E = \frac{U_0^2}{c_p A_0}\left(\frac{U_w}{U_0}\right)^{(4-\nu_0)/2}$ is the local Eckert number and $k_1^* = \frac{k_0 U_w}{\gamma l}$ is local viscoelastic parameter, $Sc = \frac{\gamma}{D}$ is the Schmidt number and $r_c = \frac{2k_c l}{U_w}$ is the local first-order chemical reaction rate.

Temperature and concentration boundary conditions of Eq. (4.1) take the following non-dimensional form:

$$\begin{aligned} &\theta(0) = 1, \qquad \theta(\infty) = 0 \\ &\phi(0) = 1, \qquad \phi(\infty) = 0 \end{aligned} \tag{4.5}$$

Now we solve Eqs. (4.3) and (4.4) by using zeroth-order approximations of $f(\eta)$ and $f_\eta(\eta)$ given by (3.12) and (3.11) respectively by introducing

$$\xi = -\frac{Pr}{S_0^2} \operatorname{Exp}(-S_0\eta), \qquad \varsigma = -\frac{Sc}{S_0^2} \operatorname{Exp}(-S_0\eta) \tag{4.6}$$

Substituting the transformations of Eq. (4.6) in Eqs. (4.3)–(4.5) we obtain the following two-point boundary value problems:

$$\begin{aligned} &\xi\theta_{\xi\xi} + (1 - Pr^* - \xi)\theta_\xi + \nu_0\theta \\ &\quad = -\frac{-ES_0^2}{Pr^*}\xi\left[1 - \delta\frac{k_1^*}{2}\left(1 - \frac{2\xi}{Pr^*}\right)\right] \\ &\zeta\phi_{\zeta\zeta} + (1 - Sc^* - \zeta)\phi_\zeta + \left(\nu_1 - \frac{Sc^*}{\zeta} r_c\right)\phi = 0 \end{aligned} \tag{4.7}$$

$$\begin{aligned} &\theta(\xi) = 1 \quad \text{at } \xi = -Pr^* \\ &\theta(\xi) = 0 \quad \text{at } \xi = 0 \end{aligned} \tag{4.8}$$

$$\begin{aligned} &\phi(\varsigma) = 1 \quad \text{at } \zeta = -Sc^* \\ &\phi(\zeta) = 0 \quad \text{at } \zeta = 0 \end{aligned} \tag{4.9}$$

where $Pr^* = Pr/S_0^2$ is the modified Prandtl number and $Sc^* = Sc/S_0^2$ is the modified Schmidt number respectively.

Now we assume the solution of Eq. (4.7) in the form

$$\theta(\xi) = \theta_c(\xi) + \theta_p(\xi) \tag{4.10}$$

where $\theta_c(\xi)$ is the complementary solution and $\theta_p(\xi)$ stands for particular integral. Making use of the boundary conditions (4.8) we obtain complementary solution of Eq. (4.7) in the following form of confluent hypergeometric function

$$\theta_c(\xi) = A_1 \xi^{Pr^*} M(Pr^* - \nu_0, Pr^* + 1, \xi) \tag{4.10a}$$

Closed form particular solution of Eq. (4.7) exists for the case $E \neq 0$ only if

$$\nu_0 = 2\left[1 - \frac{\delta k_1^*(Pr^* - 2)}{Pr^*(1 - k_1^*/2)}\right]$$

and the closed from particular solution is

$$\theta_p(\xi) = \frac{ES_0^2(1 - \delta k_1^*/2)}{2Pr^*(Pr^* - 2)}\xi^2 \tag{4.11}$$

Making use of the boundary conditions of Eq. (4.8) and rewriting the solution in variable η, we get

$$\begin{aligned} \theta(\eta) &= \theta_c(\eta) + \theta_p(\eta) \\ &= \frac{(1 - C_1)e^{-S_0 Pr^*\eta} M(Pr^* - \nu_0, Pr^* + 1, -Pr^* e^{-S_0\eta})}{M(Pr^* - \nu_0, Pr^* + 1, -Pr^*)} \\ &\quad + C_1 e^{-2S_0\eta} \end{aligned} \tag{4.11a}$$

Here

$$C_1 = \frac{-ES_0^2 Pr^*}{2(2 - Pr^*)}\left(1 - \frac{\delta k_1^*}{2}\right)$$

$$\phi(\eta) = e^{-S_0\eta(\frac{Sc^* + d_0}{2})} \frac{M[\frac{Sc^* + d_0}{2} - \nu_1, 1 + d_0, -Sc^* e^{-S_0\eta}]}{M[\frac{Sc^* + d_0}{2} - \nu_1, 1 + d_0, -Sc^*]}$$

where $Sc^* = Sc/S_0^2$, $d_0 = \sqrt{Sc^{*2} + 4r_c Sc^*}$ and Kummer's function M [31] is defined by

$$\begin{aligned} &M(a_0, b_0, z) = 1 + \sum_{n=1}^{\infty} \frac{(a_0)_n z^n}{(b_0)_n n!} \\ &(a_0)_n = a_0(a_0 + 1)(a_0 + 2)\cdots(a_0 + n - 1) \\ &(b_0)_n = b_0(b_0 + 1)(b_0 + 2)\cdots(b_0 + n - 1) \end{aligned} \tag{4.12}$$

The value of ν_1 could be chosen independently as Eq. (4.7) is a homogeneous differential equation.

Dimensionless wall temperature gradient $\theta_\eta(0)$ is obtained as

$$\begin{aligned} \theta_\eta(0) &= (1 - C_1)S_0 Pr^* \\ &\quad \times \left[\left(\frac{Pr^* - \nu_0}{Pr^* + 1}\right)\frac{M(Pr^* - \nu_0 + 1, Pr^* + 2, -Pr^*)}{M(Pr^* - \nu_0, Pr^* + 1, -Pr^*)} \right. \\ &\quad \left. - 1\right] - 2C_1 S_0 \end{aligned} \tag{4.13}$$

Dimensionless wall concentration gradient $\phi_\eta(0)$ is obtained as

$$\phi_\eta(0) = \frac{Sc^* + d_0 - 2\nu_1}{2(1+d_0)} \frac{Sc}{S_0} \frac{M(\frac{Sc^*+d_0}{2} - \nu_1 + 1, d_0 + 2, -Sc^*)}{M(\frac{Sc^*+d_0}{2} - \nu_1, d_0 + 1, -Sc^*)} - \frac{(Sc^* + d_0)S_0}{2} \tag{4.14}$$

Case B: Prescribed exponential order surface heat flux (PEHF) and surface mass flux (PEMF). In this heating process we employ the following prescribed exponential order surface heat flux and exponential order surface mass flux boundary conditions:

$$\begin{aligned}
&-k\left(\frac{\partial T}{\partial y}\right)_w = E_2 \operatorname{Exp}\left(\frac{\nu_2+1}{2l}\right)x \quad \text{at } y = 0\\
&T \to T_\infty \quad \text{as } y \to \infty\\
&-D\left(\frac{\partial C}{\partial y}\right)_w = E_3 \operatorname{Exp}\left(\frac{\nu_3+1}{2l}\right)x \quad \text{at } y = 0\\
&C \to C_\infty \quad \text{as } y \to \infty
\end{aligned} \tag{4.15}$$

where ν_2 and E_2 are the parameters of temperature distribution and E_3 and ν_3 are the parameters of concentration distribution on the stretching surface.

In order to obtain local similar solution for temperature and concentration we define dimensionless variables for temperature and concentration as

$$\begin{aligned}
g(\eta) &= \frac{T - T_\infty}{\frac{E_2}{k}\sqrt{\frac{2\gamma l}{U_0}} \operatorname{Exp}(\frac{\nu_2 x}{2l})}\\
h(\eta) &= \frac{C - C_\infty}{\frac{E_3}{D}\sqrt{\frac{2\gamma l}{U_0}} \operatorname{Exp}(\frac{\nu_3 x}{2l})}
\end{aligned} \tag{4.16}$$

With these dimensionless variables and Eqs. (3.1)–(3.3), the temperature equation (2.3) and concentration equation (2.4) take the form:

$$\begin{aligned}
&g_{\eta\eta} + Pr(f g_\eta - \nu_2 f_\eta g)\\
&\quad = -Pr\, E\left[f_{\eta\eta}^2 - \frac{\delta k_1^*}{2} f_{\eta\eta}(3 f_\eta f_{\eta\eta} - f f_{\eta\eta\eta})\right]
\end{aligned} \tag{4.17}$$

$$h_{\eta\eta} + Sc\, f h_\eta - Sc(\nu_3 f_\eta + r_c)h = 0 \tag{4.18}$$

where $E = \frac{U_0^2 k}{c_p E_2 \sqrt{2\gamma l/U_0}} \left(\frac{U_w}{U_0}\right)^{(4-\nu_2)/2}$ is the local Eckert number, $Pr = \frac{\gamma}{S_0}$ is the Prandtl number, $\mathrm{Sc} = \frac{\gamma}{D}$ is Schmidt number and $r_c = \frac{2k_c l}{U_w}$ is local first-order chemical reaction rate.

Boundary conditions on non-dimensional temperature and concentration are

$$\begin{aligned}
&g_\eta(0) = -1, \qquad h_\eta(0) = -1\\
&g(\infty) = 0, \qquad h(\infty) = 0
\end{aligned} \tag{4.19}$$

Eq. (4.17) is the same form as Eq. (4.3). However, the first boundary condition of Eq. (4.5) differs with that of Eq. (4.19). Following the same procedure as described in the PEST case and making use of the boundary conditions of Eq. (4.19) we derive the solutions for $g(\eta)$ and $h(\eta)$ in the following form of confluent hyper geometric function:

$$g(\eta) = C_2 e^{-S_0 Pr^* \eta} M\left[Pr^* - \nu_2, Pr^* + 1, -Pr^* e^{-S_0\eta}\right] + C_1 e^{-2S_0\eta} \tag{4.20}$$

where

$$C_2 = (1 - 2C_1 S_0)\left[S_0 Pr^* M(Pr^* - \nu_2, Pr^* + 1, -Pr^*) - \frac{(Pr^* - \nu_2)}{(Pr^* + 1)} Pr^* S_0 M(Pr^* - \nu_2 + 1, Pr^* + 2, -Pr^*)\right]^{-1}$$

and C_1 is given by Eq. (4.11b):

$$\begin{aligned}
h(\eta) = {} & e^{-S_0\eta(\frac{Sc^*+d_0}{2})} M\left[\frac{Sc^*+d_0}{2} - \nu_3, 1 + d_0, -Sc^* e^{-S_0\eta}\right]\\
&\times \left\{S_0\left(\frac{Sc^*+d_0}{2}\right) M\left[\frac{Sc^*+d_0-2\nu_3}{2}, 1 + d_0, -Sc^*\right]\right.\\
&- S_0 Sc^* \frac{Sc^*+d_0-2\nu_3}{2(1+d_0)}\\
&\left.\times M\left[\frac{Sc^*+d_0-2\nu_3}{2} + 1, 2 + d_0, -Sc^*\right]\right\}^{-1}
\end{aligned} \tag{4.21}$$

The expression for ν_2 in the case $E \neq 0$ takes from

$$\nu_2 = 2\left[1 - \frac{\delta k_1^*(Pr^* - 2)}{Pr^*(1 - k_1^*/2)}\right]$$

Dimensionless wall temperature $g(0)$ and wall concentration $h(0)$ are obtained as

$$g(0) = C_2 M(Pr^* - \nu_2, Pr^* + 1, -Pr^*) + C_1$$

$$\begin{aligned}
h(0) = {} & M\left[\frac{Sc^*+d_0}{2} - \nu_3, 1 + d_0, -Sc^*\right]\\
&\times \left\{S_0\left(\frac{Sc^*+d_0}{2}\right) M\left[\frac{Sc^*+d_0-2\nu_3}{2}, 1 + d_0, -Sc^*\right]\right.\\
&- S_0 Sc^* \frac{Sc^*+d_0-2\nu_3}{2(1+d_0)}\\
&\left.\times M\left[\frac{Sc^*+d_0-2\nu_3}{2} + 1, 2 + d_0, -Sc^*\right]\right\}^{-1}
\end{aligned} \tag{4.22}$$

The value ν_3 could be chosen independently as Eq. (4.18) is a homogeneous differential equation.

The expressions for dimensional wall temperature and wall concentration are

$$\begin{aligned}
T_w &= T_\infty + \frac{E_2}{k}\sqrt{\frac{2\gamma l}{U_0}} \operatorname{Exp}\left(\frac{\nu_2 x}{2l}\right) g(0)\\
C_w &= C_\infty + \frac{E_3}{D}\sqrt{\frac{2\gamma l}{U_0}} \operatorname{Exp}\left(\frac{\nu_3 x}{2l}\right) h(0)
\end{aligned} \tag{4.23}$$

5. Results and discussion

Momentum, heat and mass transfers in a boundary layer viscoelastic fluid flow over an exponentially stretching impermeable sheet have been examined in this paper. The highly non-linear partial differential equation for momentum transfer characterising flow phenomena has been converted to a non-linear quasi-ordinary differential equation by applying suitable pseudo-similarity transformations. Sequential solutions of the transformed momentum equation are obtained by solving the non-linear Riccati type equation analytically. The zeroth-order approximate solution for dimensionless stream function f has been obtained analytically which satisfies all the boundary conditions. The graphical behaviour of the functions f and f_η and nature of the prescribed boundary conditions has promoted us to assume the zeroth-order solution of the form of equation (3.12). In order to solve the fourth-order quasi-linear equation (3.5) we must have fourth boundary conditions. However, we have been given three boundary conditions of (3.6) which is one less than the required one to obtain solution uniquely. Therefore we generate fourth boundary condition on f by substituting the boundary conditions of (3.6) in the basic equation (3.5) and obtained

$$f_{\eta\eta\eta}(0) = \frac{4 - 3k_1^* f_{\eta\eta}^2(0)}{2(1 - 3k_1^*)} \tag{5.1}$$

Now fourth-order equation (3.5) has been solved numerically using the given three boundary conditions of Eq. (3.6) and the fourth boundary condition (4.1). First-order approximate solution of f also can be derived analytically in the form of confluent hypergeometric functions. However numerical solutions for f and f_η, using Runge–Kutta fourth-order method with shooting, match very well with the solution of zeroth-order in the region close to the boundary sheet (Fig. 1). Hence, we consider zeroth-order approximate solutions of f and obtain the exact analytical solutions of the heat transfer equation in the form of confluent hypergeometric functions. All these solutions involve an exponential dependence of (i) the pseudo-similarity variable (ii) stretching velocity and (iii) wall temperature distribution and wall concentration distribution on the coordinate x along the direction of stretching.

Results are plotted graphically for typical choice of physical parameters in Figs. 3–6 and Tables 1–2. The values of ν_1 and ν_3 are chosen as 2. Fig. 3 demonstrates the graph of non-dimensional local skin-friction parameter C_f vs. viscoelastic parameter k_1^* for different values of local Reynolds number Re. From this figure we observe that the increase of non-dimensional local viscoelastic parameter k_1^* leads to the decrease of local skin-friction parameter C_f. This is due to the fact that elastic property in viscoelastic fluid reduces the frictional force. This result may have great significance in polymer proceeding industry as the choice of higher-order viscoelastic fluid would reduce the power consumption for stretching the boundary sheet. We obtain the similar effect of local Reynolds number on the local skin-friction coefficient as in this case the reduction of viscosity of the fluid results in the decrease of frictional force or drag force.

The effect of Prandtl number Pr on heat transfer process may be analysed from Figs. 4(a) and 4(b) in PEST and PEHF cases respectively. These graphs reveal that the increase of Prandtl number Pr results in the decrease of temperature distribution at a particular point of the flow region. This is because there would be a decrease of the thermal boundary layer thickness with the increase of values of Prandtl number Pr. The increase of Prandtl number means slow rate of thermal diffusion. However, in the presence of viscous dissipation the effect of increasing the values of Prandtl number Pr is to increase temperature distribution near the boundary and decrease everywhere away from the boundary in PEST case. The effect of elastic deformation of the fluid is to decease temperature distribution throughout the flow region. It is obvious that the wall temperature distribution is at unity on the wall in PEST case for all the values of Pr and E. However, it may be other than the unity in the PEHF case due to adiabatic temperature boundary condition. The results of PEHF cases are qualitatively similar to that of PEST case but quantitatively they are different. The graphs reveal that the effect of increasing the values of local Eckert number E is to increase temperature distribution in the flow region in both the cases of PEST and PEHF. This behaviour of temperature enhancement occurs as heat energy is stored in the fluid due to frictional heating. The effect of elastic deformation is accounted by the term $\delta = 1$ in the energy equation. It is seen that the effect of elastic deformation is to reduce the temperature throughout the flow field which is the consequence of elastic property of the fluid.

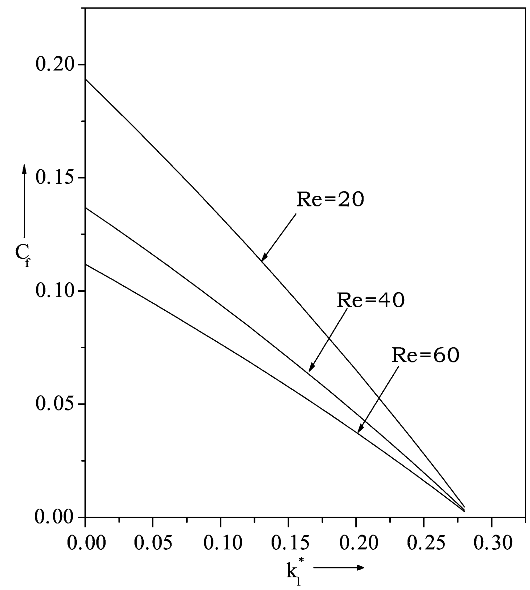
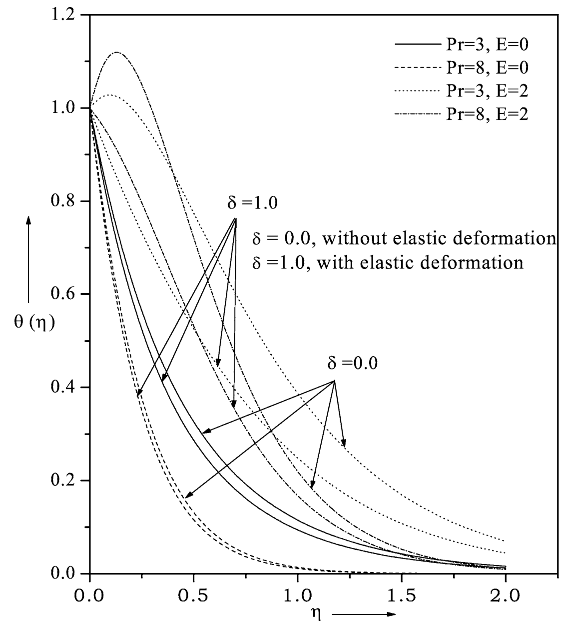
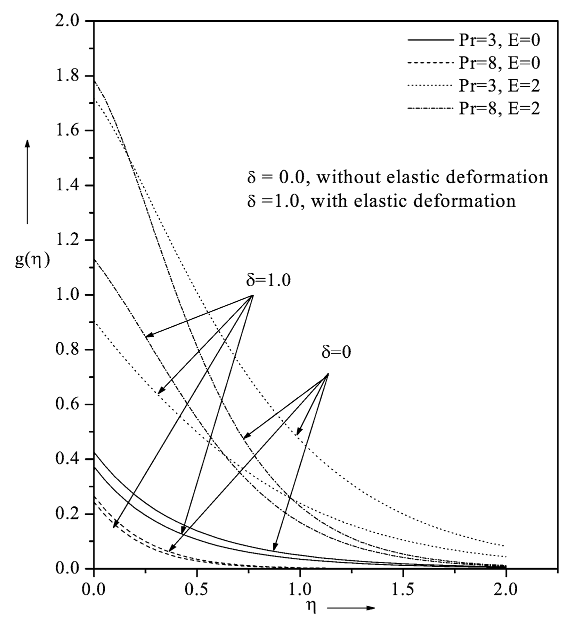


Fig. 3. Graph of local skin-friction parameter C_f vs. visco-elastic parameter k_1^* for different values of local Reynolds number Re.

The effect of Schmidt number Sc on mass transfer process may be analysed from Figs. 5(a) and 5(b) for the case of prescribed exponential order surface concentration (PESC) and prescribed exponential order surface mass flux (PEMF), respectively. Figs. 5(a) and 5(b) show that the increase of value of Schmidt number Sc results in the decrease of concentration distribution as a result of decrease of the concentration boundary layer thickness with the increased values of Sc. Similar to the results of temperature distribution we notice that the wall concentration distribution is at unity on the wall in PESC case and is other then the unity in PEMF case. However, the effect of increasing the local viscoelastic parameter k_1^* is seen to increase the concentration distribution in the flow region. This is because the concentration boundary layer increases as a re-

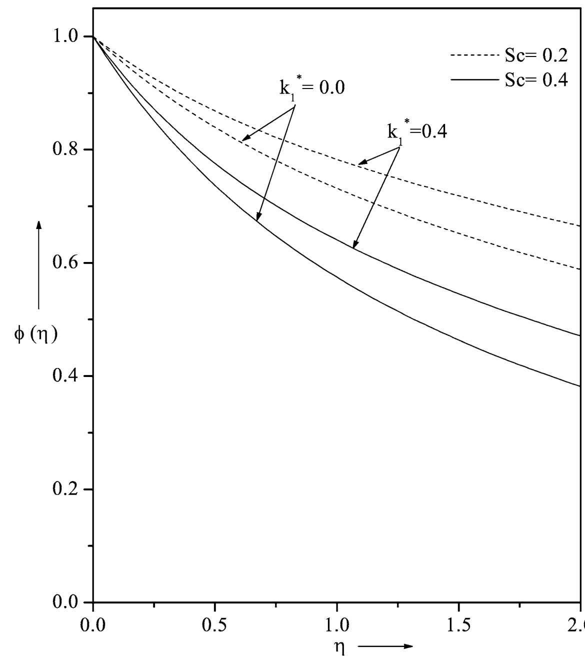


(a)

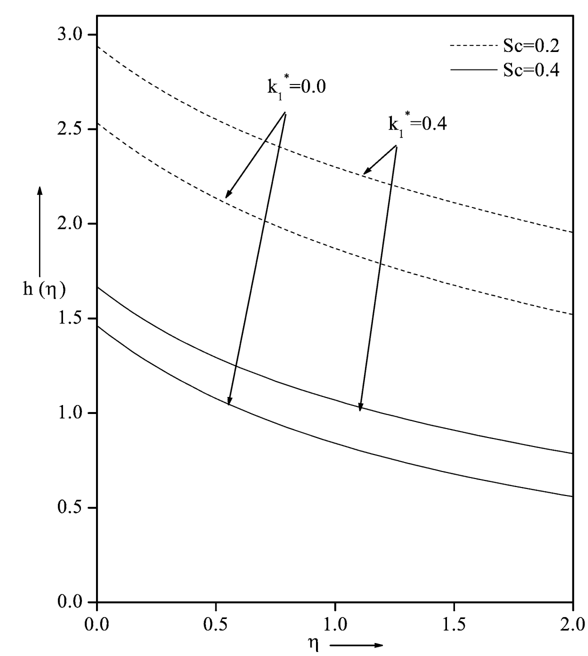


(b)

Fig. 4. (a) Dimensionless temperature profile $\theta(\eta)$ for various values of Prandtl number Pr and Eckert number E when $k_1^* = 0.4$ in PEST case. (b) Dimensionless temperature profile $g(\eta)$ for various values of Prandtl number Pr and Eckert number E when $k_1^* = 0.4$ in PEHF case.



(a)



(b)

Fig. 5. (a) Dimensionless concentration profile for various values of k_1^* and Sc when $r_c = 0.01$ in PESC case. (b) Dimensionless concentration profile for various values of k_1^* and Sc when $r_c = 0.01$ in PEMF case.

sult of viscoelasticity of the fluid. The results of PEMF case are qualitatively similar to that of PESC case but they differ quantitatively.

Figs. 6(a) and 6(b) demonstrate that the effect of first-order chemical conversion rate on the species concentration in PESC and PEMF cases. In conformity with the reality in both the cases the increase of local chemical reaction rate r_c is to decrease concentration throughout the concentration boundary layer. This is due to the fact that the conversion of the species takes place as a result of chemical reaction and thereby reduces the concentration in the boundary layer.

Numerical values of wall temperature gradient $-\theta_\eta(0)$ in PEST case for different values of Prandtl number Pr, local Eckert number E and local viscoelastic parameter k_1^* are recorded in the Table 1. The value of ν_0 is chosen as 2.67 for the case when $E = 0$. The tabulated values reveal that the increase of the values of Prandtl number Pr for $E = 0$ results in the increase of the values of wall temperature gradient $-\theta_\eta(0)$. Table shows that there is no effect of elastic deformation term (accounting by $\delta = 1$) on the temperature gradient as the local Eckert number

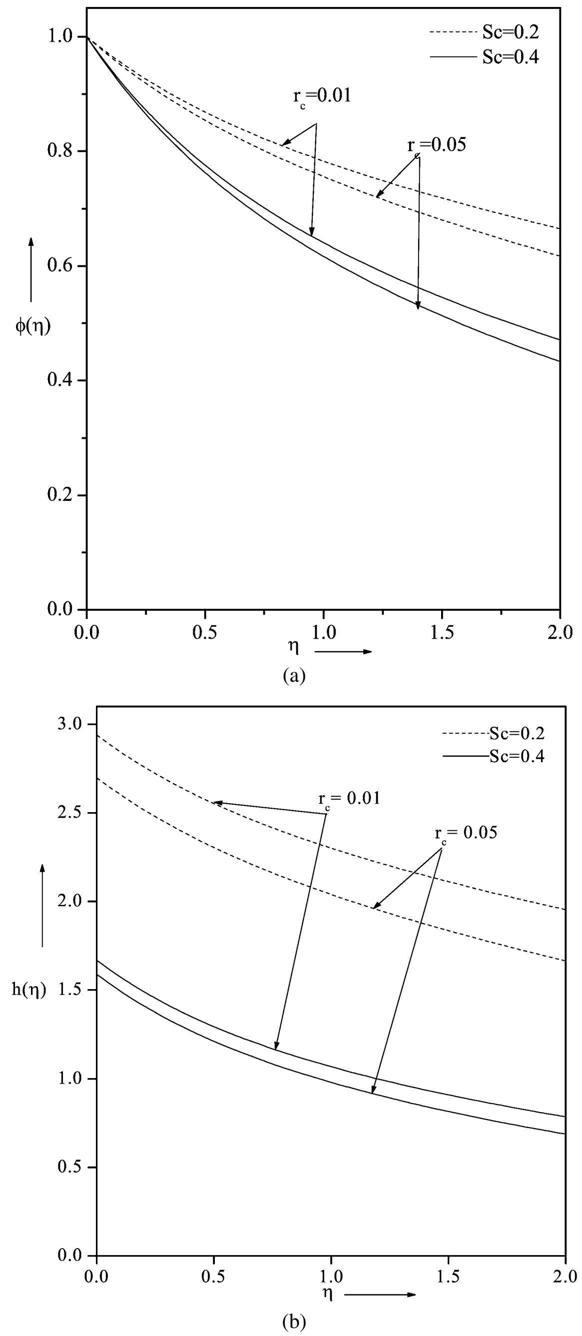


Fig. 6. (a) Dimensionless concentration profile for various values of *Sc* and r_c when $k_1^* = 0.04$ in PESC case. (b) Dimensionless concentration profile for various values of *Sc* and r_c when $k_1^* = 0.04$ in PEMF case.

$E = 0$ implies there is no generation of heat due to elastic deformation as well as viscous dissipation of the fluid. By increasing the values of the local viscoelastic parameter k_1^* we notice that the wall temperature gradient $-\theta_\eta(0)$ is decreased. The effect of viscous dissipation ($E \neq 0$ and $\delta = 0$) is to reduce the wall temperature gradient $-\theta_\eta(0)$ significantly. The effect of elastic deformation and viscous dissipation ($\delta = 1$ and $E \neq 0$) on the temperature is zero in case of viscous fluid. However, the effect of elastic deformation is to reverse the heat transfer direction from the stretching sheet by changing the sign of the heat transfer gradient in case of viscoelastic fluid. In presence the of viscous dissipation and elastic deformation ($E = 2$ and $\delta = 1$) the effect of increasing the values of k_1^* is to change the sign of the values of $-\theta_\eta(0)$ for large values of Prandtl number $Pr = 8$. Hence, we can reverse the direction of heat transfer by changing the values of Prandtl number Pr.

Table 1
Wall temperature gradient $-\theta_\eta(0)$ in PEST case for different values of Prandtl number Pr, Eckert number E and viscoelastic parameter k_1^*

k_1^*	Pr	E	$-\theta_\eta(0)$ $\delta = 0.0$	$-\theta_\eta(0)$ $\delta = 1.0$
10^{-9}	3	0	2.78	2.78
0.4			2.68	2.68
10^{-9}	8		4.75	4.75
0.4			4.65	4.65
10^{-9}	3	2	0.36	0.36
0.4			−0.69	1.25
10^{-9}	8		−0.01	−0.01
0.4			−2.22	0.50

Table 2
Wall temperature $g(0)$ in PEHF case for different values of Prandtl number Pr, Eckert number E and viscoelastic parameter k_1^*

k_1^*	Pr	E	$g(0)$ $\delta = 0.0$	$g(0)$ $\delta = 1.0$
10^{-9}	3	0	0.369	0.369
0.4			0.372	0.372
10^{-9}	8		0.210	0.210
0.4			0.215	0.215
10^{-9}	3	2	1.259	1.259
0.4			1.718	0.905
10^{-9}	8		1.239	1.239
0.4			1.783	1.131

Table 2 is plotted for the different values of Prandtl number Pr, local Eckert number E and local viscoelastic parameter k_1^* for wall temperature $g(0)$ in PEHF case. The value of ν_2 is chosen as 2.67 for the case when $E = 0$. Analysis of the tabular results shows the similar features of heat transfer for various values of Pr, E and k_1^*. As the value of Prandtl number Pr increases for $E = 0$ the wall temperature $g(0)$ decreases and increase of the values of the local viscoelastic parameter k_1^* leads to the increase of wall temperature $g(0)$. From this table we notices that the effect of elastic deformation is to decrease wall temperature only when $E \neq 0$, $k_1^* \neq 0$.

6. Conclusions

The effect of various physical parameters on momentum, heat and mass phenomena in a viscoelastic fluid flow over an exponentially stretching impermeable sheet has been analysed. Here the highly non-linear differential equation characterizing the flow has been transformed into a quasi-ordinary differential equation by applying suitable transformations and sequential solutions of the transformed momentum equation are obtained analytically by solving the non-linear Riccati-type equation repeatedly. Solutions for heat and mass transfer equations are derived in the form of confluent hypergeometric functions for both the cases (i) prescribed exponential order surface temperature (PEST) and (ii) prescribed exponential order boundary heat flux (PEHF). Expressions are also obtained for dimensionless local skin-friction coefficients C_f. The derived solutions involve an exponential dependence of stretching velocity, prescribed surface temperature and concentration and prescribed surface heat flux and mass flux on the flow directional coordinate.

The important findings of the graphical analysis of the results of the present problem are:

(1) The effect of increasing the values of local viscoelastic parameter k_1^* is to decrease the local skin-friction parameter C_f and the effect of local Reynolds number is also to decrease local skin-friction coefficient C_f.
(2) In the presence of viscous dissipation the effect of increasing the values of Prandtl number Pr is to increase temperature distribution near the boundary and decrease everywhere away from the boundary.
(3) The effect of elastic deformation of the fluid is to decease temperature distribution throughout the flow region.
(4) In the presence of viscous dissipation and elastic deformation one can reverse the direction of heat transfer by suitably changing values of Prandtl number Pr. However, in absence of viscous dissipation the effect of Prandtl number Pr is to increase the heat transfer rate significantly.

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